Two loop and all loop finite 4-metrics

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Abstract

In pure Einstein theory, Ricci flat Lorentzian 4-metrics of Petrov types III or N have vanishing counter terms up to and including two loops. Moreover for pp-waves and type-N spacetimes of Kundt's class which admit a non-twisting, non expanding, null congruence all possible invariants formed from the Weyl tensor and its covariant derivatives vanish. Thus these Lorentzian metrics suffer no quantum corrections to all loop orders. By contrast for complete non-singular Riemannian metrics the two loop counter term vanishes only if the metric is flat.

Solutions of classical field equations for which the counter terms required to regularize quantum fluctuations vanish are of particular importance because they offer insights into the behaviour of the full quantum theory. Conversely, classical solutions for which quantum corrections do not vanish afford little or no insight into the strongly quantum regime since quantum corrections are expected to be large. Thus in gravity theories, in plane wave spacetimes, which are of Petrov type N, all counter terms vanish on shell and so they suffer no quantum corrections [1, 2], This corresponds to the fact that the graviton remains massless in the quantum theory and presumably remains a physically valid concept no matter how large quantum effects are.

Another important class of examples occur in supersymmetric theories when one has a supersymmetric, so-called BPS, solution of the classical equations. In four-dimensional supergravity theories, examples of BPS solutions are provided by positive definite metrics whose curvature is self-dual or anti-self dual, or equivalently $SU(2) \equiv Sp(1)$ holonomy. An interesting question is whether there are non-trivial positive definite metrics which are not supersymmetric and for which quantum corrections nevertheless vanish in pure gravity. One purpose of the present note is to show that, subject to regularity, there are none.

The coefficients of quantum corrections to Ricci flat solutions of Einstein's theory of gravity in four dimensions have been calculated up to two loops. However it is widely believed that there are non-vanishing terms to arbitrarily high loop order. The one loop term counter term is proportional to quadratic invariant of the curvature tensor $I_2 = R_{abcd}R^{abcd}$ the integrand of the Gauss-Bonnet theorem. If the metric is Lorentzian and of Petrov types III or N, then this invariant vanishes [10]. However, if the background has a positive definite metric, then the counter term I_2 is positive definite and can only vanish if the background is flat. However because it is a total derivative, one might think that this fact may not be important, and could perhaps be ignored. Even if this point is conceded one still has two consider higher looops. Let us therefore turn to the two loop term which is proportional to the cubic invariant of the curvature tensor $I_3 = R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab}$ [7, 8]. If the background is Lorentzian and of Petrov types III or N then this invariant also vanishes [10]. Indeed in a Ricci flat spacetime of Petrov types III or N all invariants formed solely by contracting products of the Riemann tensor using the metric must necessarily vanish. This property, which holds only for these Petrov types, is clear from the expression for the Weyl Spinor C_{ABCD} in terms of the principle spinors κ^A and ι^A. For type N

$$C_{ABCD} = \iota_A\iota_B\iota_C\iota_D. \tag{1}$$

and for type III

$$C_{ABCD} = \iota_A\iota_B\iota_{(C}\kappa_{D)}, \tag{2}$$

and no non-vanishing contractions are possible.

Even if the background metric is positive definite, a cubic invariant cannot be positive definite and so it makes sense to ask: are there non-singular backgrounds $\{M, g\}$ for which it vanishes? Note that there is no analogue of Petrov type III or N for positive definite metrics [9]. However it is possible to give a similar algebraic characterization of the vanishing of the cubic invariant. The self-dual and anti-self-dual parts of the Weyl tensor determines two symmetric trace-free symmetric matrices $D^{\pm}_{ij}$ say. The invariant I_3 is proportional to

$$\mathrm{tr}(D^+)^3 + \mathrm{tr}(D^-)^3. \tag{3}$$

A short calculation reveals that I_3 will vanish if and only if both D^+ and D^- have a zero eigen-value.

Remarkably, the answer to the question asked above is provided, almost word for word, by a rather old result of Lichnerowicz [3, 4] (see also [6, 5]). Lichnerowicz derives an identity, valid in all dimensions if the Ricci tensor vanishes, which reads

$$\nabla^2 I_2 = 2\nabla_e R_{abcd}\nabla^e R^{abcd} + K. \tag{4}$$

where the scalar (as opposed to pseudo-scalar) K is a cubic invariant of the curvature tensor. In four dimensions K must obviously be a multiple of I_3 since if the Ricci tensor vanishes that is the only non-vanishing scalar cubic invariant.

One may now integrate (4) over the manifold M. The left hand side gives a boundary term, which obviously vanishes if M is closed and which one may check will certainly vanish if M is ALE or ALF. We deduce that if $I_3 = 0$ then

$$\nabla_e R_{abcd} = 0. \tag{5}$$

In fact (5) says that M is locally at least a symmetric space. However it is clear that one may deduce more. Since

$$\nabla_e I_2 = 0, \tag{6}$$

it follows in the ALF or ALE case that $I_2 = 0$ everywhere, since it vanishes at infinity. But if $I_2 = 0$ the metric must be locally flat. If M is closed, then we need a more complicated argument. However the conclusion is the same: the metric must be locally flat [3].

Thus even self-dual metrics, are not exact and would receive quantum corrections in pure Einstein theory at two loops. By contrast, non-trivial vacuum Lorentzian metrics of Petrov type III and N, which are not plane waves or pp-waves certainly exist [11]. In the case of type N vacuum spacetimes, it is known [12] that provided they admit a non-expanding, non-twisting geodesic null congruence, then all invariants formed from the Weyl tensor and it's covariant derivatives vanish. It seems that this class of solutions, which belong to Kundt's class [11], do not correspond to our conventional idea of a graviton since gravitons are usually taken to be described by pp-waves. In view of the vanishing of all quantum corrections it would seem worthwhile investigating these metrics further. Since pp-waves admit a covariantly constant spinor, they have a holonomy group which is not the full Lorentz group. An interesting question is whether the other all loop finite metrics in Kundt's class also have a reduced holonomy group.

If one drops the non-expanding condition but retains the non-twisting condition then there are non-vanishing invariants. For example, in [12] the invariant

$$C^{abcd;ef} C_{amcn;ef} C^{lmrn;st} C_{lbrd;st} \tag{7}$$

is shown not to vanish. The situation with respect to type III spacetimes is not known to the author.

We close with the remark that it would be interesting to extend these results when a cosmological term is present, or to higher dimensions. In the latter case Lichnerowicz's identity holds in all dimensions. However the structure of the counter terms and possible invariants will be different.

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